Anticipating Quantum Stochastic Integrals for Basic Quantum Martingales (Mathematical aspects of quantum fields and related topics)

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数理解析研究所講究録 第2123巻 2019年 29-46

Anticipating Quantum Stochastic Integrals for Basic Quantum Martingales

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1 Introduction

Since the quantum stochastic integrals of adapted quantum stochastic processes have been introduced by Hudson and Parthasarathy [10] as a quantum extension of the Itô (stochastic) integral, the quantum stochastic calculus has been studied extensively with wide applications (see [28, 32]).

The Hudson-Parthasarathy quantum stochastic integrals has been extended to the quantum stochastic integrals of nonadapted quantum stochastic processes by Belavkin [3], Lindsay [24] and Attal & Lindsay [2]. Since then the nonadapted quantum stochastic integral has been studied systematically in terms of quantum stochastic gradients by Ji & Obata [16, 18]. Based on the quantum white noise theory [12], the notion of quantum white noise derivatives has been introduced by Ji & Obata (see [14, 15, 16, 17, 19, 20]). The explicit formulas [16] of integrands for quantum stochastic integral representation of quantum martingales [11] has been derived in terms of the quantum white noise derivatives. Also, the notion of quantum stochastic gradients [18] has been introduced based on the notion of the quantum white noise derivatives. Recently, Ji & Sinha [21] studied the quantum stochastic integrals for quadratic quantum noises.

On the other hand, based on the white noise theory [8, 22, 30] introduced by Hida, Kuo & Russek [23] studied anticipating (classical) stochastic integrals by applying the quantum decomposition of a Brownian motion.

In this paper, we study some regularity properties of the quantum Hitsuda–Skorohod integrals as anticipating quantum stochastic integrals. Also, motivated by the results in [23], we discuss new types of anticipating quantum stochastic integrals in terms of pointwisely defined quantum white noise derivatives.

2 Admissible Generalized Operators

2.1 Admissible Rigging of Fock Space

We now review a construction of admissible rigging of Fock space which provides the basic structure of this paper. Let $H = L^2(\mathbb{R}_+, dt)$ be the Hilbert space of complex valued square integrable functions on $\mathbb{R}_+ = [0, \infty)$ with respect to the Lebesque measure dt and let $\Gamma(H)$ be the Fock space over H defined by

$$\Gamma(H) = \left\{ \phi = (f_n)_{n=0}^{\infty} \,;\, f_n \in H^{\widehat{\otimes} n}, \ \sum_{n=0}^{\infty} n! |f_n|^2 < \infty \right\},$$

where $H^{\widehat{\otimes} n}$ is the n-fold symmetric tensor product of H and $|\cdot|$ is the Hilbertian norm on H and $H^{\widehat{\otimes} n}$. For $p \geq 0$, we set

$$\mathcal{G}_p = \left\{ \phi = (f_n)_{n=0}^{\infty} \in \Gamma(H) \,;\, \|\phi\|_p^2 = \sum_{n=0}^{\infty} n! e^{2pn} |f_n|^2 < \infty \right\}$$

and $\mathcal{G}_{-p}$ to be the completion of $\Gamma(H)$ with respect to the norm $\|\cdot\|_{-p}$ defined by

$$\|\phi\|_{-p}^2 = \sum_{n=0}^{\infty} n! e^{-2pn} |f_n|^2.$$

Then $\{\mathcal{G}_p \,;\, p \in \mathbb{R}\}$ forms a chain of weighted Fock spaces and so we have

$$\mathcal{G} = \operatorname*{proj\,lim}_{p \to \infty} \mathcal{G}_p \subset \mathcal{G}_p \subset \mathcal{G}_0 = \Gamma(H) \subset \mathcal{G}_{-p} \subset \mathcal{G}^* \cong \operatorname*{ind\,lim}_{p \to \infty} \mathcal{G}_{-p}$$

for $p \geq 0$, where the strong dual space $\Gamma(H)^*$ of $\Gamma(H)$ is identified with $\Gamma(H)$, and the strong dual space $\mathcal{G}^*$ of $\mathcal{G}$ is topologically isomorphic to the inductive limit space $\operatorname{ind\,lim}_{p \to \infty} \mathcal{G}_{-p}$. The canonical $\mathbb{C}$-bilinear form $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ on $\mathcal{G}^* \times \mathcal{G}$ takes the form:

$$\langle\!\langle \Phi, \phi \rangle\!\rangle = \sum_{n=0}^{\infty} n! \langle f_n, g_n \rangle, \qquad \Phi = (f_n) \in \mathcal{G}^*, \quad \phi = (g_n) \in \mathcal{G},$$

where $\langle f_n, g_n \rangle$ is the canonical $\mathbb{C}$-bilinear form on $H^{\otimes n} \times H^{\otimes n}$. Note that $\mathcal{G}$ is a countable Hilbert space but not necessarily a nuclear space. An element in $\mathcal{G}$ is said to be *admissible* or *regular*.

Remark 2.1 Let $E_{\mathbb{R}} \subset H_{\mathbb{R}} \subset E_{\mathbb{R}}^*$ be a Gelfand triple, i.e. $E_{\mathbb{R}}$ is a nuclear space, where $H_{\mathbb{R}} = L^2_{\mathbb{R}}(\mathbb{R}_+, dt)$ is the Hilbert space of real valued square integrable functions on $\mathbb{R}_+$ with respect to dt. Then for the standard Gaussian measure μ on $E_{\mathbb{R}}^*$ characterized by

$$\int_{E_{\mathbb{R}}^*} e^{i\langle x, \xi \rangle} \, d\mu(x) = e^{-\frac{1}{2}|\xi|^2}, \qquad \xi \in E_{\mathbb{R}},$$

where $\langle \cdot, \cdot \rangle$ is the canonical bilinear form on $E_{\mathbb{R}}^* \times E_{\mathbb{R}}$ again, by the Wiener-Itô-Segal isomorphism, $\Gamma(H)$ is unitarily equivalent with the Hilbert space $L^2(E_{\mathbb{R}}^*, \mu)$ of complex valued square integrable functions on $E_{\mathbb{R}}^*$ with respect to the Gaussian measure μ. In this sense, the elements of $\mathcal{G}$ are considered as admissible Gaussian functionals. The spaces $\mathcal{G}$ and $\mathcal{G}^*$ were introduced by Belavkin [3] and have appeared along with classical and quantum stochastic analysis, see e.g., [1, 4, 7, 11, 13, 14, 24, 25, 26, 33, 34].

2.2 Multiplications of Admissible Gaussian Functionals

Let $\phi = (f_n), \psi = (g_n) \in \mathcal{G}$ be given. Suppose that $f_n = 0$ and $g_m = 0$ except for finite numbers of n and m. Then the *Wiener product* (or *pointwise multiplication*) $\phi\psi \in \mathcal{G}$ of ϕ and ψ is defined by

$$\phi\psi = (h_n), \qquad h_n = \sum_{l+m=n} \sum_{k=0}^{\infty} k! \binom{l+k}{k} \binom{m+k}{k} f_{l+k} \widehat{\otimes}_k g_{m+k}, \tag{2.1}$$

where $f_{l+k} \widehat{\otimes}_k g_{m+k}$ is the k-contraction of f_{l+k} and g_{m+k}, see [30].

The following lemma is useful to study the continuities of Wiener product of admissible Gaussian functionals and similar estimates can be found in [25] (see also [30, 33]).

Lemma 2.2 *Let $\phi = (f_n), \psi = (g_n) \in \mathcal{G}$ be given. Suppose that $f_n = 0$ and $g_m = 0$ except for a finite numbers of n and m. Then for any $p, r, s \in \mathbb{R}$ with $r + s > 0$ and*

$$(n+1) \left(\frac{e^{(s-3r)/2} + e^{(r-3s)/2}}{e^{-2p}(r+s)} \right)^n \leq c^n \tag{2.2}$$

for some $0 < c < 1$, it holds that

$$|\!|\!| \phi\psi |\!|\!|_p^2 \leq \frac{1}{1-c} |\!|\!| \phi |\!|\!|_r^2 \, |\!|\!| \psi |\!|\!|_s^2 . \tag{2.3}$$

Proof. For given h_n as in (2.1), we obtain that

$$\begin{aligned} n!|h_n|^2 &= n! \left(\sum_{l+m=n} \sum_{k=0}^{\infty} k! \binom{l+k}{k} \binom{m+k}{k} |f_{l+k}||g_{m+k}| \right)^2 \\ &\leq n! \left(\sum_{l+m=n} \sum_{k=0}^{\infty} M_{l,m,k} \sqrt{(l+k)!} e^{r(l+k)} |f_{l+k}| \sqrt{(m+k)!} e^{s(m+k)} |g_{m+k}| \right)^2 , \end{aligned} \tag{2.4}$$

where

$$M_{l,m,k} = \frac{e^{-rl-sm}}{l!m!} \frac{\sqrt{(l+k)!(m+k)!}}{k!} e^{-(r+s)k} \leq \frac{e^{-rl-sm}}{l!m!} \sqrt{C_{l,m;r+s}},$$

where

$$C_{l,m;q} = \sup_{n \geq 0} \left\{ \frac{(l+n)!}{n!} \frac{(n+m)!}{n!} e^{-2qn} \right\} \leq e^q l^l m^m \left(\frac{e^{q/2}}{eq} \right)^{l+m} < \infty \tag{2.5}$$

for $q > 0$ (see e.g., [30]: Section 4.1). Therefore, for any $r \in \mathbb{R}$ and $s \in \mathbb{R}$ with $r + s > 0$, from (2.4), by applying Cauchy-Schwarz inequality we obtain that

$$\begin{aligned} n!|h_n|^2 &\leq n! \left(\sum_{l+m=n} \frac{e^{-rl-sm}}{l!m!} \sqrt{C_{l,m;r+s}} \right)^2 |\!|\!| \phi |\!|\!|_r^2 \, |\!|\!| \psi |\!|\!|_s^2 \\ &\leq \left(\sum_{l+m=n} \frac{\sqrt{n!}}{l!m!} e^{-rl-sm} \sqrt{C_{l,m;r+s}} \right)^2 |\!|\!| \phi |\!|\!|_r^2 \, |\!|\!| \psi |\!|\!|_s^2 . \end{aligned} \tag{2.6}$$

By applying a simple inequality $n^n \le e^n n!$, from (2.5) we see that

$$\sqrt{C_{l,m;r+s}} \le e^{(r+s)/2}\sqrt{l^l m^m}\left(\frac{e^{(r+s)/2}}{e(r+s)}\right)^{(l+m)/2} \le e^{(r+s)/2}\sqrt{l!m!}\left(\frac{e^{(r+s)/2}}{r+s}\right)^{(l+m)/2}$$

Therefore, for any $r \in \mathbb{R}$ and $s \in \mathbb{R}$ with $e^{r+s} \ge 2$, from (2.6) we obtain that

$$\begin{aligned} n!|h_n|^2 &\le \left(\sum_{l+m=n}\frac{\sqrt{n!}}{l!m!}e^{-rl-sm}\sqrt{C_{l,m;r+s}}\right)^2 \| \phi \|_r^2 \| \psi \|_s^2 \\ &\le e^{r+s}\left(\sum_{l+m=n}\frac{\sqrt{n!}}{\sqrt{l!m!}}e^{-rl-sm}\left(\frac{e^{(r+s)/2}}{r+s}\right)^{(l+m)/2}\right)^2 \| \phi \|_r^2 \| \psi \|_s^2 \\ &\le (n+1)\left(\sum_{l+m=n}\frac{n!}{l!m!}\left(\frac{e^{(s-3r)/2}}{r+s}\right)^l\left(\frac{e^{(r-3s)/2}}{r+s}\right)^m\right) \| \phi \|_r^2 \| \psi \|_s^2 \\ &= (n+1)\left(\frac{e^{(s-3r)/2}+e^{(r-3s)/2}}{r+s}\right)^n \| \phi \|_r^2 \| \psi \|_s^2 . \end{aligned} \tag{2.7}$$

Therefore, from (2.1) and (2.7) we obtain that

$$\begin{aligned} \| \phi\psi \|_p^2 &= \sum_{n=0}^{\infty} n! e^{2pn}|h_n|^2 \le \left[\sum_{n=0}^{\infty}(n+1)\left(\frac{e^{(s-3r)/2}+e^{(r-3s)/2}}{e^{-2p}(r+s)}\right)^n\right] \| \phi \|_r^2 \| \psi \|_s^2 \\ &\le \frac{1}{1-c} \| \phi \|_r^2 \| \psi \|_s^2 \end{aligned} \tag{2.8}$$

for some $0 < c < 1$ satisfying (2.3), which gives the proof. □

The following two theorem are obvious consequences of Lemma 2.2.

Theorem 2.3 ([33]) *The Wiener product of admissible Gaussian functionals is continuous from $\mathcal{G} \times \mathcal{G}$ (equipped with the product topology) onto $\mathcal{G}$. In particular, $\mathcal{G}$ is an algebra with respect to the Wiener product.*

Theorem 2.4 *The Wiener product of admissible white noise functionals is continuous from $\mathcal{G}^* \times \mathcal{G}$ (equipped with the product topology) onto $\mathcal{G}^*$.*

Let $\phi = (f_n), \psi = (g_n) \in \mathcal{G}$ be given. Suppose that $f_n = 0$ and $g_m = 0$ except for finite numbers of n and m. Then the *Wick product* (or *normal-ordered product*) $\phi \diamond \psi$ of ϕ and ψ is defined by

$$\phi \diamond \psi = (k_n), \qquad k_n = \sum_{l+m=n} f_l \widehat{\otimes} g_m, \tag{2.9}$$

see [6, 8, 22]

The following lemma is useful to study the continuities of Wick product of admissible Gaussian functionals and similar estimates can be found in [33].

Lemma 2.5 *Let $\phi = (f_n), \psi = (g_n) \in \mathcal{G}$ be given. Suppose that $f_n = 0$ and $g_m = 0$ except for a finite numbers of n and m. Then for any $p, r, s \in \mathbb{R}$ satisfying that*

$$e^{2(p-r)} + e^{2(p-s)} < 1, \tag{2.10}$$

it holds that

$$\| \phi \diamond \psi \|_p^2 \leq \| \phi \|_r^2 \| \psi \|_s^2 . \tag{2.11}$$

Proof. For given k_n as in (2.9), we obtain that

$$\begin{aligned} n!|k_n|^2 &= n! \left(\sum_{l+m=n} |f_l||g_m| \right)^2 \\ &\leq n! \left(\sum_{l+m=n} \frac{e^{-2rl-2sm}}{l!m!} \right) \left(\sum_{l+m=n} l!e^{2rl}|f_l|^2 m!e^{2sm}|g_m|^2 \right) \\ &\leq \left(e^{-2r} + e^{-2s}\right)^n \left(\sum_{l+m=n} l!e^{2rl}|f_l|^2 m!e^{2sm}|g_m|^2 \right), \end{aligned} \tag{2.12}$$

Therefore, for any $p, r, s \in \mathbb{R}$ satisfying (2.10), from (2.12) we obtain that

$$\begin{aligned} \| \phi \diamond \psi \|_p^2 = \sum_{n=0}^{\infty} n!e^{2pn}|k_n|^2 &\leq \sum_{n=0}^{\infty} \left(e^{2(p-r)} + e^{2(p-s)}\right)^n \left(\sum_{l+m=n} l!e^{2rl}|f_l|^2 m!e^{2sm}|g_m|^2 \right) \\ &\leq \| \phi \|_r^2 \| \psi \|_s^2 , \end{aligned}$$

which gives the proof. □

The following theorem is an obvious consequence of Lemma 2.5.

Theorem 2.6 ([33]) *The Wick product is continuous from $\mathcal{G} \times \mathcal{G}$ (equipped with the product topology) onto $\mathcal{G}$, and from $\mathcal{G}^* \times \mathcal{G}^*$ onto $\mathcal{G}^*$. In particular, $\mathcal{G}$ and $\mathcal{G}^*$ are algebras under the Wick product.*

3 Admissible Generalized Operators

We denote by $\mathcal{L}(\mathfrak{X}, \mathfrak{Y})$ the space of all continuous linear operators from a locally convex space $\mathfrak{X}$ into another locally convex space $\mathfrak{Y}$ equipped with the topology of bounded convergence. An operator in $\mathcal{L}(\mathcal{G}, \mathcal{G}^*)$ is called an *admissible generalized operator* [14] or simply *admissible operator*.

3.1 Integral Kernel Operators

Let l, m be non-negative integers. Let $K_{l,m} \in \mathcal{L}(H^{\otimes m}, H^{\otimes l})$ and $\Phi = (f_n)_{n=0}^{\infty} \in \mathcal{G}^*$. For each $n \geq 0$, we put

$$g_{l+n} = \frac{(n+m)!}{n!} (K_{l,m} \otimes I^{\otimes n} f_{n+m})_{\mathrm{sym}}. \tag{3.1}$$

Then from Lemma 4.1 in [11], for any $p \in \mathbb{R}$ and $q > 0$, we obtain that

$$\begin{aligned}\sum_{n=0}^{\infty}(l+n)!e^{2p(l+n)}|g_{l+n}|^2 &\leq \|K_{l,m}\|^2 \sum_{n=0}^{\infty}(n+m)!\,\frac{(l+n)!}{n!}\frac{(n+m)!}{n!}e^{2p(l+n)}|f_{n+m}|^2 \\ &\leq \|K_{l,m}\|^2 e^{2(pl-(p+q)m)}C_{l,m;q}\,\|\!|\,\phi\,\|\!|_{p+q}^2\,, \qquad (3.2)\end{aligned}$$

where $\|K_{l,m}\|$ is the operator norm and $C_{l,m;q}$ is given as in (2.5). Therefore, we define an linear operator $\Xi_{l,m}(K_{l,m})$ on $\mathcal{G}^*$ by

$$\Xi_{l,m}(K_{l,m})\Phi = (g_{l+n})_{n=0}^{\infty}\,, \qquad \Phi = (f_n)_{n=0}^{\infty} \in \mathcal{G}^*, \qquad (3.3)$$

where g_{l+n} is given as in (3.1). Then for any $p \in \mathbb{R}$ and $q > 0$ it holds that

$$\|\!|\,\Xi_{l,m}(K_{l,m})\Phi\,\|\!|_p \leq \|K_{l,m}\|e^{(pl-(p+q)m)}\sqrt{C_{l,m;q}}\,\|\!|\,\Phi\,\|\!|_{p+q}\,, \qquad \Phi \in \mathcal{G}^*, \qquad (3.4)$$

which implies that $\Xi_{l,m}(K_{l,m}) \in \mathcal{L}(\mathcal{G}_{p+q}, \mathcal{G}_p)$. The operator $\Xi_{l,m}(K_{l,m})$ is called the *integral kernel operator* with kernel $K_{l,m}$ (see [13, 9, 22, 30]).

Now the following theorem is obvious.

Theorem 3.1 ([11]) *Let l, m be non-negative integers and let $K_{l,m} \in \mathcal{L}(H^{\otimes m}, H^{\otimes l})$. Then it holds that*

$$\Xi_{l,m}(K_{l,m}) \in \mathcal{L}(\mathcal{G}, \mathcal{G}) \cap \mathcal{L}(\mathcal{G}^*, \mathcal{G}^*).$$

Let $\eta \in H$ and let $K_\eta \in \mathcal{L}(H, \mathbb{C})$ be defined by $K_\eta(f) = \langle \eta,\, f\rangle$ for any $f \in H$. For simple notation, we identify $\eta = K_\eta = K_\eta^*$, where K_η^* is the adjoint operator of K_η with respect to the canonical bilinear form $\langle \cdot,\, \cdot\rangle$, i.e., $K_\eta^*(a) = a\eta$ for all $a \in \mathbb{C}$. Then the *annihilation operator* $a(\eta)$ and the *creation operator* $a^*(\eta)$ associated with η are defined by $a(\eta) = \Xi_{0,1}(\eta)$ and $a^*(\eta) = \Xi_{1,0}(\eta)$, respectively, and then from Theorem 3.1, it holds that

$$a(\eta), a^*(\eta) \in \mathcal{L}(\mathcal{G}, \mathcal{G}) \cap \mathcal{L}(\mathcal{G}^*, \mathcal{G}^*).$$

It is straightforward to verify the canonical commutation relation:

$$[a(\xi), a(\eta)] = 0, \quad [a^*(\xi), a^*(\eta)] = 0, \quad [a(\xi), a^*(\eta)] = \int_{\mathbb{R}_+} \xi(t)\eta(t)dt = \langle \xi, \eta\rangle \qquad (3.5)$$

for $\xi, \eta \in H$.

The exponential vector ϕ_ξ associated with $\xi \in H$ is defined by $\phi_\xi = (\xi^{\otimes n}/n!)_{n=0}^{\infty}$. Then $\{\phi_\xi\,;\, \xi \in H\}$ spans a dense subspace of $\mathcal{G}$.

Proposition 3.2 ([14, 5]) *Let $\zeta \in H$ be given. Then it holds that*

$$a(\zeta)(\Phi\psi) = (a(\zeta)\Phi)\,\psi + \Phi\,(a(\zeta)\psi)\,, \qquad \Phi \in \mathcal{G}^*, \quad \psi \in \mathcal{G}, \qquad (3.6)$$

$$a(\zeta)(\Phi \diamond \Psi) = (a(\zeta)\Phi) \diamond \Psi + \Phi \diamond (a(\zeta)\Psi)\,, \qquad \Phi, \Psi \in \mathcal{G}^*. \qquad (3.7)$$

PROOF. (i) For any $\xi, \eta \in H$, we obtain that

$$\begin{aligned}a(\zeta)(\phi_\xi\phi_\eta) &= a(\zeta)(\phi_{\xi+\eta})e^{\langle\xi,\,\eta\rangle} = \langle \zeta,\, \xi+\eta\rangle\,\phi_{\xi+\eta}e^{\langle\xi,\,\eta\rangle} = \langle\zeta,\,\xi+\eta\rangle\,\phi_\xi\phi_\eta \\ &= (a(\zeta)\phi_\xi)\,\phi_\eta + \phi_\xi\,(a(\zeta)\phi_\eta)\,.\end{aligned}$$

Therefore, by the continuity property $a(\zeta) \in \mathcal{L}(\mathcal{G}, \mathcal{G}) \cap \mathcal{L}(\mathcal{G}^*, \mathcal{G}^*)$ and the fact that exponential vectors span a dense subspace of $\mathcal{G}$ and $\mathcal{G}^*$, we complete the proof.

(ii) The proof is similar to the proof of (i). In fact, we obtain that

$$\begin{aligned} a(\zeta)(\phi_\xi \diamond \phi_\eta) &= a(\zeta)(\phi_{\xi+\eta}) = \langle \zeta, \xi + \eta \rangle \, \phi_{\xi+\eta} = \langle \zeta, \xi + \eta \rangle \, \phi_\xi \diamond \phi_\eta \\ &= (a(\zeta)\phi_\xi) \diamond \phi_\eta + \phi_\xi \diamond (a(\zeta)\phi_\eta) \, . \end{aligned}$$

□

Let $K \in \mathcal{L}(H, H)$. Then from Theorem 3.1, it holds that

$$\Lambda(K) := \Xi_{1,1}(K) \in \mathcal{L}(\mathcal{G}, \mathcal{G}) \cap \mathcal{L}(\mathcal{G}^*, \mathcal{G}^*).$$

The operator $\Lambda(K)$ is called the *conservation operator.* for any $p \in \mathbb{R}$ and $q > 0$, from (3.4) we obtain that

$$\| \Lambda(K)\Phi \|_p \le e^{-q} \sqrt{C_{1,1;q}} \|K\| \, \| \Phi \|_{p+q} \, , \qquad \Phi \in \mathcal{G}^*. \tag{3.8}$$

3.2 Multiplication Operators

Theorem 3.3 *For any $\Phi \in \mathcal{G}^*$ and $\phi, \psi \in \mathcal{G}$, it holds that*

$$\langle\langle \Phi\phi, \, \psi \rangle\rangle = \langle\langle \Phi, \, \phi\psi \rangle\rangle \, . \tag{3.9}$$

Proof. For given $\Phi = (F_n) \in \mathcal{G}^*$ and any $\xi, \eta \in H$, from (2.1) we obtain that

$$\Phi\phi_\xi = \left(\sum_{l+m=n} \sum_{k=0}^{\infty} \binom{l+k}{k} \left(F_{l+k} \widehat{\otimes}_k \xi^{\otimes k} \right) \otimes \frac{\xi^{\otimes m}}{m!} \right)_{n=0}^{\infty}$$

and

$$\begin{aligned} \langle\langle \Phi\phi_\xi, \, \phi_\eta \rangle\rangle &= \sum_{n=0}^{\infty} \left\langle \sum_{m=0}^{n} \sum_{k=0}^{\infty} \binom{n-m+k}{k} \left(F_{n-m+k} \widehat{\otimes}_k \xi^{\otimes k} \right) \otimes \frac{\xi^{\otimes m}}{m!}, \, \eta^{\otimes n} \right\rangle \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left\langle \sum_{k=0}^{\infty} \binom{n+k}{k} \left(F_{n+k} \widehat{\otimes}_k \xi^{\otimes k} \right) \otimes \frac{\xi^{\otimes m}}{m!}, \, \eta^{\otimes (n+m)} \right\rangle \\ &= e^{\langle \xi, \, \eta \rangle} \sum_{l=0}^{\infty} \left\langle F_l, \, \sum_{n+k=l} \binom{n+k}{k} \eta^{\otimes n} \widehat{\otimes} \xi^{\otimes k} \right\rangle \\ &= e^{\langle \xi, \, \eta \rangle} \, \langle\langle \Phi, \, \phi_{\xi+\eta} \rangle\rangle \\ &= \langle\langle \Phi, \, \phi_\eta \phi_\eta \rangle\rangle \, . \end{aligned}$$

Since the exponential vectors span a dense subspace of $\mathcal{G}$, by the continuity of the Wiener product (see Theorems 2.3 and 2.4), the proof is immediate. □

Let $\Phi \in \mathcal{G}^*$ be given. Then we consider the Wiener multiplication operator $M_\Phi : \mathcal{G} \to \mathcal{G}^*$ and then from (3.9), $\phi, \psi \in \mathcal{G}$, we obtain that it holds that

$$\langle\langle M_\Phi \phi, \, \psi \rangle\rangle = \langle\langle \Phi\phi, \, \psi \rangle\rangle = \langle\langle \Phi, \, \phi\psi \rangle\rangle \, .$$

Theorem 3.4 ([30]) *For each $\zeta \in H$, $X_\zeta = (0, \zeta, 0, \cdots) \in \mathcal{G}$ as a Wiener multiplication operator is represented as the sum of $a(\zeta)$ and $a^*(\zeta)$, i.e,*

$$X_\zeta = a(\zeta) + a^*(\zeta), \tag{3.10}$$

which is called the quantum decomposition of X_η.

Proof. Since $X_\zeta = (0, \zeta, 0, \cdots) \in \mathcal{G}$, from (3.9) we obtain that

$$\begin{aligned}\langle X_\zeta \phi_\xi, \, \phi_\eta \rangle &= \langle X_\zeta, \, \phi_\xi \phi_\eta \rangle = \langle X_\zeta, \, \phi_{\xi+\eta} \rangle \, e^{\langle \xi, \, \eta \rangle} = \langle \zeta, \, \xi + \eta \rangle \, e^{\langle \xi, \, \eta \rangle} \\ &= \langle (a(\zeta) + a^*(\zeta)) \, \phi_\xi, \, \phi_\eta \rangle \,,\end{aligned}$$

which gives the proof. □

For each $t \geq 0$, put $B_t = X_{\mathbf{1}_{[0,t]}}$. Then $\{B_t\}_{t \geq 0}$ becomes a Brownian motion which is called a *realization of Brownian motion* and so from Theorem 3.4 we have the following quantum decomposition of Brownian motion:

$$B_t = a(\mathbf{1}_{[0,t]}) + a^*(\mathbf{1}_{[0,t]}), \qquad t \geq 0. \tag{3.11}$$

Remark 3.5 The operators $\mathcal{L}(\mathcal{G}, \mathcal{G}^*)$ on admissible Gaussian functionals play an essential role in the study of quantum martingales and integral representations [11, 14, 16, 17].

4 Quantum White Noise Derivatives

In this section, we briefly review some basic properties of quantum white noise derivatives [14, 15, 16, 17, 18, 20].

4.1 Annihilation and Creation Derivatives

For any admissible operator $\Xi \in \mathcal{L}(\mathcal{G}, \mathcal{G}^*)$ and $\zeta \in H$, from Theorem 3.1 the commutators

$$[a(\zeta), \Xi] = a(\zeta)\Xi - \Xi a(\zeta), \qquad -[a^*(\zeta), \Xi] = \Xi a^*(\zeta) - a^*(\zeta)\Xi$$

are well defined as compositions of admissible operators, i.e., belong to $\mathcal{L}(\mathcal{G}, \mathcal{G}^*)$. We define

$$D_\zeta^+ \Xi = [a(\zeta), \Xi], \qquad D_\zeta^- \Xi = -[a^*(\zeta), \Xi].$$

These are called the *creation derivative* and *annihilation derivative* of Ξ, respectively. Both together are referred to as the *quantum white noise derivatives* (*qwn-derivatives* for brevity) of Ξ. By the definitions, it is obvious that

$$\begin{aligned}\left(D_\zeta^+ \Xi^*\right)^* &= ([a(\zeta), \Xi^*])^* = (a(\zeta)\Xi^* - \Xi^* a(\zeta))^* = \Xi a^*(\zeta) - a^*(\zeta)\Xi \\ &= D_\zeta^- \Xi.\end{aligned} \tag{4.1}$$

For each admissible operator $\Xi \in \mathcal{L}(\mathcal{G}_p, \mathcal{G}_q)$, we operator norm of Ξ is denoted by $\|\Xi\|_{p;q}$.

Theorem 4.1 ([14]) *Let $\zeta \in H$ be given. Then $D_\zeta^\pm$ are continuous linear operators from $\mathcal{L}(\mathcal{G}, \mathcal{G}^*)$ itself.*

Proof. Suppose that $\Xi \in \mathcal{L}(\mathcal{G}_p, \mathcal{G}_q)$. Then for any $r > 0$, by applying (3.4), we obtain that

$$\begin{aligned}\left\| D_\zeta^+ \Xi \right\|_{p-r;q+r} &= \| [a(\zeta), \Xi] \|_{p-r;q+r} = \| a(\zeta)\Xi - \Xi a(\zeta) \|_{p-r;q+r} \\ &\le \| a(\zeta) \|_{q;q+r} \| \Xi \|_{p;q} + \| \Xi \|_{p;q} \| a(\zeta) \|_{p-r;p},\end{aligned}$$

which implies that D_ζ^+ is a continuous linear operator on $\mathcal{L}(\mathcal{G}, \mathcal{G}^*)$. Similarly, we see that D_ζ^- is a continuous linear operator on $\mathcal{L}(\mathcal{G}, \mathcal{G}^*)$. □

Proposition 4.2 *For each $\zeta \in H$ and $\Phi \in \mathcal{G}^*$, it holds that*

$$\left(D_\zeta^+ M_\Phi \right) \phi_0 = \left(D_\zeta^- M_\Phi \right) \phi_0 = a(\zeta)\Phi.$$

Proof. We obtain that

$$\begin{aligned}\left(D_\zeta^+ M_\Phi \right) \phi_0 &= \left(a(\zeta) M_\Phi - M_\Phi a(\zeta) \right) \phi_0 = a(\zeta)\Phi, \\ \left(D_\zeta^- M_\Phi \right) \phi_0 &= \left(M_\Phi a^*(\zeta) - a^*(\zeta) M_\Phi \right) \phi_0 = \Phi X_\zeta - a^*(\zeta)\Phi = a(\zeta)\Phi,\end{aligned}$$

where we used the quantum decomposition as $\Phi X_\zeta = X_\zeta \Phi = (a(\zeta) + a^*(\zeta))\Phi$. □

4.2 Pointwise QWN-Derivatives

Let $\phi = (f_n) \in \mathcal{G}$ and $t \in \mathbb{R}_+$ be given. Suppose that $f_n = 0$ except for a finite number of n. We define

$$D_t \phi := (n f_n(t, \cdot))_{n=1}^\infty,$$

where $f_n(t, \cdot) \in H^{\widehat{\otimes}(n-1)}$, and then D is called the *classical stochastic gradient*. The classical stochastic gradient is denoted by ∇ in some literatures see [8, 16, 18, 22, 29]. We now extend the domain of D to the space $\mathcal{G}^*$.

Lemma 4.3 ([16]) *For any $p \in \mathbb{R}$ and $r > 0$ we have*

$$\| D\phi \|_{L^2(\mathbb{R}_+, \mathcal{G}_{-p-r})}^2 = \int_{\mathbb{R}_+} \| D\phi(t) \|_{-p-r}^2 \, dt \le K(p,r) \| \phi \|_{-p}^2 , \qquad \phi \in \mathcal{G}, \tag{4.2}$$

where $K(p,r) = \sup_n (n+1) e^{2p-2rn} < \infty$. In particular, the classical stochastic gradient

$$D : \mathcal{G}_{-p} \to L^2(\mathbb{R}_+, \mathcal{G}_{-p-r}) \cong L^2(\mathbb{R}_+) \otimes \mathcal{G}_{-p-r} \tag{4.3}$$

is a continuous linear map.

Proof. For each $\phi = (f_n)_{n=0}^\infty \in \mathcal{G}$ consisting of continuous functions f_n on $\mathbb{R}_+^n$, we have $D\phi(t) = ((n+1) f_{n+1}(t, \cdot))_{n=0}^\infty$, where the right-hand side has a pointwise meaning. Then we obtain that

$$\begin{aligned}\int_{\mathbb{R}_+} \| D\phi(t) \|_{-p-r}^2 \, dt &= \sum_{n=0}^\infty n! e^{-2(p+r)n} \int_{\mathbb{R}_+} |(n+1) f_{n+1}(t, \cdot)|_0^2 dt \\ &= \sum_{n=0}^\infty (n+1) e^{2p-2rn} \times (n+1)! e^{-2p(n+1)} |f_{n+1}|_0^2 \\ &\le K(p,r) \| \phi \|_{-p}^2 ,\end{aligned}$$

which implies the proof of (4.2). □

Put

$$L^2(\mathbb{R},\mathcal{G}) := \operatorname*{proj\,lim}_{p\to\infty} L^2(\mathbb{R},\mathcal{G}_p) \cong \operatorname*{proj\,lim}_{p\to\infty} L^2(\mathbb{R})\otimes\mathcal{G}_p\,,$$
$$L^2(\mathbb{R}_+,\mathcal{G}^*) := \operatorname*{ind\,lim}_{p\to\infty} L^2(\mathbb{R}_+,\mathcal{G}_{-p}) \cong \operatorname*{ind\,lim}_{p\to\infty} L^2(\mathbb{R})\otimes\mathcal{G}_{-p}.$$

Then by Lemma 4.3, the classical stochastic gradient D is a continuous linear map from $\mathcal{G}$ into $L^2(\mathbb{R}_+,\mathcal{G})$ and from $\mathcal{G}^*$ into $L^2(\mathbb{R}_+,\mathcal{G}^*)$.

We see from (4.3) that $D\Phi(t)$ has a meaning as $\mathcal{G}_{-p-r}$-valued L^2-function in $t\in\mathbb{R}_+$. Given $\zeta\in L^2(\mathbb{R}_+)$, the linear map $\mathcal{G}_{p+r}\ni\psi\mapsto\langle\!\langle D\Phi,\,\zeta\otimes\psi\rangle\!\rangle$ is continuous. Therefore there exists a unique $\Psi\in\mathcal{G}_{-p-r}$ such that

$$\langle\!\langle D\Phi,\,\zeta\otimes\psi\rangle\!\rangle = \langle\!\langle \Psi,\,\psi\rangle\!\rangle, \qquad \psi\in\mathcal{G}_{p+r}.$$

It is reasonable to write

$$\Psi = \int_{\mathbb{R}_+}\zeta(t)D\Phi(t)\,dt.$$

As is easily seen, the Schwarz inequality holds:

$$\left\|\!\left\| \int_{\mathbb{R}_+}\zeta(t)D\Phi(t)\,dt\right\|\!\right\|_{-p-r} \le |\zeta|_0 \,\|\!|\, D\Phi\,\|\!|_{L^2(\mathbb{R}_+,\mathcal{G}_{-p-r})}\,, \tag{4.4}$$

which implies that the map

$$\mathcal{G}_{-p}\ni\Phi\mapsto\int_{\mathbb{R}_+}\zeta(t)D\Phi(t)\,dt\in\mathcal{G}_{-(p+r)}$$

is continuous. On the other hand, for any $\xi\in H$, we obtain that

$$\begin{aligned}\int_{\mathbb{R}_+}\zeta(t)D\phi_\xi(t)\,dt &= \int_{\mathbb{R}_+}\zeta(t)a_t\phi_\xi\,dt = \int_{\mathbb{R}_+}\zeta(t)\xi(t)\phi_\xi\,dt = \langle\zeta,\xi\rangle\phi_\xi\\ &= a(\zeta)\phi_\xi.\end{aligned}$$

Therefore, we obtain that

$$\int_{\mathbb{R}_+}\zeta(t)D\Phi(t)\,dt = a(\zeta)\Phi, \qquad \Phi\in\mathcal{G}^*, \tag{4.5}$$

see [16].

Remark 4.4 The space $\mathcal{G}^*$ as a domain of the classical gradient D appeared in Aase–Øksendal–Privault–Ubøe [1]. For a standard domain see e.g., Kuo [22], Malliavin [27], Nualart [29].

Let $\Xi\in\mathcal{L}(\mathcal{G}_p,\mathcal{G}_q)$ for some $p,q\in\mathbb{R}$. Then for any $r>0$ and $\phi\in\mathcal{G}$, from (4.2) we obtain that

$$\begin{aligned}\int_{\mathbb{R}_+}\|\!|\,\Xi D_t\phi\,\|\!|_q^2\,dt &\le \int_{\mathbb{R}_+}\|\!|\,\Xi\,\|\!|_{p;q}^2\,\|\!|\,D_t\phi\,\|\!|_p^2\,dt\\ &\le K(-p,r)\,\|\!|\,\Xi\,\|\!|_{p;q}^2\,\|\!|\,\phi\,\|\!|_{p+r}^2\,,\end{aligned}$$

which implies that

$$\int_{\mathbb{R}_+} \| \Xi D_t \|_{p+r;q}^2 \, dt \le K(-p,r) \| \Xi \|_{p;q}^2 ,$$

and so the map

$$\mathcal{L}(\mathcal{G}_p, \mathcal{G}_q) \ni \Xi \longmapsto \Xi D \in L^2(\mathbb{R}_+, \mathcal{L}(\mathcal{G}_{p+r}, \mathcal{G}_q)$$

is continuous. Similarly, we obtain that

$$\int_{\mathbb{R}_+} \| D_t \Xi \phi \|_{q-r}^2 \, dt \le K(-q,r) \| \Xi \phi \|_q^2 \le K(-q,r) \| \Xi \|_{p;q}^2 \| \phi \|_p^2 ,$$

which implies that

$$\int_{\mathbb{R}_+} \| \Xi D_t \|_{p+r;q}^2 \, dt \le K(-p,r) \| \Xi \|_{p;q}^2 ,$$

and so the map

$$\mathcal{L}(\mathcal{G}_p, \mathcal{G}_q) \ni \Xi \longmapsto D\Xi \in L^2(\mathbb{R}_+, \mathcal{L}(\mathcal{G}_p, \mathcal{G}_{q-r})$$

is continuous. Therefore, the *pointwise creation derivative* D_t^+ is defined by

$$D_t^+ \Xi = D_t \Xi - \Xi D_t, \qquad \Xi \in \mathcal{L}(\mathcal{G}, \mathcal{G}^*)$$

and $D_t^+ \Xi$ is an $\mathcal{L}(\mathcal{G}, \mathcal{G}^*)$-valued L^2-function in $t \in \mathbb{R}_+$. Motivated by (4.1), the *pointwise annihilation derivative* D_t^- is defined by

$$D_t^- \Xi = \left(D_t^+ \Xi^* \right)^* , \qquad \Xi \in \mathcal{L}(\mathcal{G}, \mathcal{G}^*),$$

see [16, 18]. In fact, for given $\Xi \in \mathcal{L}(\mathcal{G}, \mathcal{G}^*)$ and $\zeta \in H$, from (4.5) we obtain that

$$\int_{\mathbb{R}_+} \zeta(t) D_t^+ \Xi dt = D_\zeta^+ \Xi$$

and

$$D_\zeta^- \Xi = \left(D_\zeta^+ \Xi^* \right)^* = \int_{\mathbb{R}_+} \zeta(t) \left(D_t^+ \Xi^* \right)^* dt.$$

Proposition 4.5 *For each $t \ge 0$ and $\Phi \in \mathcal{G}^*$, it holds that*

$$\left(D_t^+ M_\Phi \right) \phi_0 = \left(D_t^- M_\Phi \right) \phi_0 = D_t \Phi.$$

PROOF. We obtain that

$$\begin{aligned} \left(D_t^+ M_\Phi \right) \phi_0 &= \left(D_t M_\Phi - M_\Phi D_t \right) \phi_0 = D_t \Phi, \\ \left(D_t^- M_\Phi \right) \phi_0 &= \left(D_t^+ M_\Phi^* \right)^* \phi_0 = \left(D_t^+ M_\Phi \right)^* \phi_0 = \left(M_{D_t \Phi} \right)^* \phi_0 = D_t \Phi, \end{aligned}$$

which gives the proof. $\square$

5 Anticipating Quantum Stochastic Integrals

For each $t \geq 0$, let $\mathcal{F}_t$ be the σ-field generated by $\{B_s\,;\, 0 \leq s \leq t\}$. A one-parameter family $\Phi = \{\Phi_t\}_{t\geq 0} \subset \mathcal{G}^*$ is called a *generalized stochastic process* [4, 11, 31] if there exists a $p \geq 0$ (independent of $t \geq 0$) such that $\Phi_t \in \mathcal{G}_{-p}$ for all $t \geq 0$ and the map $t \mapsto \Phi_t \in \mathcal{G}_{-p}$ is Borel measurable on $\mathbb{R}_+$. A generalized stochastic process $\{\Phi_t = (F_{t;n})\}_{t\geq 0}$ is said to be *adapted* (w.r.t. $\mathcal{F}_t$) if for all $t \geq 0$ and $n \geq 0$, $\mathrm{supp} F_{t;n} \subset [0,t]^n$.

A one-parameter family $\{\Xi_t\}_{t\in\mathbb{R}_+} \subset \mathcal{L}(\mathcal{G}, \mathcal{G}^*)$ is called a *quantum stochastic process*. Our approach covers a wide class of classical and quantum stochastic processes in the sense that $\mathcal{G}^*$ and $\mathcal{L}(\mathcal{G}, \mathcal{G}^*)$ involve distributions. As examples, for each $t \geq 0$, we put

$$A_t = a(\mathbf{1}_{[0,t]}), \quad A_t^* = a^*(\mathbf{1}_{[0,t]}), \quad \Lambda_t = \Xi_{1,1}(\mathbf{1}_{[0,t]}).$$

For the definition of Λ_t, the indicator function $\mathbf{1}_{[0,t]}$ is considered as a multiplication operator on H, i.e., $\mathbf{1}_{[0,t]}(\xi) = \mathbf{1}_{[0,t]}\xi =: \xi_{[0,t]}$ for any $\xi \in H$. Then for each $t \geq 0$, $A_t, A_t^*, \Lambda_t \in \mathcal{L}(\mathcal{G}, \mathcal{G}) \cap \mathcal{L}(\mathcal{G}^*, \mathcal{G}^*)$. The processes $\{A_t\}_{t\geq 0}$, $\{A_t^*\}_{t\geq 0}$ and $\{\Lambda_t\}_{t\geq 0}$ are called the *annihilation*, *creation* and *conservation* (or *gauge*) *processes*, respectively.

5.1 Quantum Hitsuda–Skorohod Integrals

In this section, we study the Hitsuda–Skorohod type quantum stochastic integrals with their regular properties.

Theorem 5.1 *Let $p, q \in \mathbb{R}$ be given and $\Xi \in L^2(\mathbb{R}_+, \mathcal{L}(\mathcal{G}_p, \mathcal{G}_q))$ be a quantum stochastic process. Then there exists an admissible operator, denoted by $\delta^-(\Xi)$, in $\mathcal{L}(\mathcal{G}_{p+r}, \mathcal{G}_q)$ for any $r > 0$ such that*

$$\delta^-(\Xi)\phi = \int_{\mathbb{R}_+} \Xi(t)(D_t\phi)\,dt \tag{5.1}$$

for any $\phi \in \mathcal{G}$.

Proof. For any $\phi \in \mathcal{G}$ and $r > 0$, by applying (4.2), we obtain that

$$\begin{aligned}
\left\| \int_{\mathbb{R}_+} \Xi(t)(D_t\phi)\,dt \right\|_q &\leq \int_{\mathbb{R}_+} \| \Xi(t) \|_{p;q} \, \| D_t\phi \|_p \,dt \\
&\leq \left(\int_{\mathbb{R}_+} \| \Xi(t) \|_{p;q}^2 \,dt \right)^{1/2} \left(\int_{\mathbb{R}_+} \| D_t\phi \|_p^2 \,dt \right)^{1/2} \\
&\leq \sqrt{K(-p,r)} \left(\int_{\mathbb{R}_+} \| \Xi(t) \|_{p;q}^2 \,dt \right)^{1/2} \| \phi \|_{p+r}^2 \,,
\end{aligned}$$

which implies that the linear operator

$$\mathcal{G}_{p+r} \ni \phi \longmapsto \int_{\mathbb{R}_+} \Xi(t)(D_t\phi)\,dt \in \mathcal{G}_q$$

is continuous. □

For given $\Xi \in L^2(\mathbb{R}_+, \mathcal{L}(\mathcal{G}_p, \mathcal{G}_q))$, the admissible operator $\delta^-(\Xi)$ satisfying (5.1) is called the *annihilation integral* of Ξ, see [3, 24, 16, 18].

Remark 5.2 Let $p, q \in \mathbb{R}$ be given and $\Xi \in L^2(\mathbb{R}_+, \mathcal{L}(\mathcal{G}_p, \mathcal{G}_q))$ be a quantum stochastic process. Then for any $\xi \in H$, we obtain that

$$\delta^-(\Xi)\phi_\xi = \int_{\mathbb{R}_+} \Xi(t)(D_t\phi_\xi)\, dt = \int_{\mathbb{R}_+} \xi(t)\Xi(t)\phi_\xi\, dt = \left(\int_{\mathbb{R}_+} \Xi(t)\, dA_t\right)\phi_\xi,$$

which implies that

$$\delta^-(\Xi) = \int_{\mathbb{R}_+} \Xi(t)\, dA_t$$

on a certain domain. Furthermore, if Ξ is adapted, then $\delta^-(\Xi)$ coincides with the annihilation integral of Hudson-Parthasarathy. For the definition of the adaptedness of quantum stochastic processes, we refer to [11]. Also, for more study on quantum Hitsuda–Skorohod integrals, we refer to [3, 24, 18].

As for a criterion for $\delta^-(\Xi)$ being a bounded operator on $\Gamma(H)$, we have the following corollary. A similar result can be found in [18].

Corollary 5.3 *For any $r > 0$ and $\Xi \in L^2(\mathbb{R}, \mathcal{L}(\mathcal{G}_{-r}, \Gamma(H)))$, the annihilation integral $\delta^-(\Xi)$ is a bounded operator on $\Gamma(H)$.*

PROOF. The proof is immediate from Theorem 5.1. □

Theorem 5.4 *Let $p, q \in \mathbb{R}$ be given and $\Xi \in L^2(\mathbb{R}_+, \mathcal{L}(\mathcal{G}_p, \mathcal{G}_q))$ be a quantum stochastic process. Then there exists an admissible operator, denoted by $\delta^+(\Xi)$, in $\mathcal{L}(\mathcal{G}_p, \mathcal{G}_{q-r})$ for any $r > 0$ such that*

$$\langle\langle \delta^+(\Xi)\phi,\, \psi\rangle\rangle = \int_{\mathbb{R}_+} \langle\langle \Xi(t)\phi,\, D_t\psi\rangle\rangle\, dt \tag{5.2}$$

for $\phi, \psi \in \mathcal{G}$.

PROOF. For any $\phi, \psi \in \mathcal{G}$ and $r > 0$, by applying (4.2), we obtain that

$$\begin{aligned}\left|\int_{\mathbb{R}_+} \langle\langle \Xi(t)\phi,\, D_t\psi\rangle\rangle\, dt\right| &\le \int_{\mathbb{R}_+} \|\, \Xi(t)\phi\,\|_q \,\|\, D_t\psi\,\|_{-q}\, dt \\ &\le \left(\int_{\mathbb{R}_+} \|\, \Xi(t)\,\|_{p;q}^2\, dt\right)^{1/2} \left(\int_{\mathbb{R}_+} \|\, D_t\psi\,\|_{-q}^2\, dt\right)^{1/2} \|\,\phi\,\|_p \\ &\le \sqrt{K(q,r)} \left(\int_{\mathbb{R}_+} \|\, \Xi(t)\,\|_{p;q}^2\, dt\right)^{1/2} \|\,\phi\,\|_p \,\|\,\psi\,\|_{-q+r},\end{aligned}$$

which implies that the bilinear form

$$\mathcal{G}_p \times \mathcal{G}_{-q+r} \ni (\phi, \psi) \longmapsto \int_{\mathbb{R}_+} \langle\langle \Xi(t)\phi,\, D_t\psi\rangle\rangle\, dt \in \mathbb{C}$$

is continuous. Therefore, there exists a unique admissible operator $\delta^+(\Xi) \in \mathcal{L}(\mathcal{G}_p, \mathcal{G}_{p-r})$ such that (5.2) holds. □

For given $\Xi \in L^2(\mathbb{R}_+, \mathcal{L}(\mathcal{G}_p, \mathcal{G}_q))$, the admissible operator $\delta^+(\Xi)$ satisfying (5.2) is called the *creation integral* of Ξ, see [3, 24, 16, 18].

As for a criterion for $\delta^+(\Xi)$ being a bounded operator on $\Gamma(H)$, we have the following corollary. A similar result can be found in [18].

Corollary 5.5 *For any $r > 0$ and $\Xi \in L^2(\mathbb{R}, \mathcal{L}(\Gamma(H), \mathcal{G}_r))$, the creation integral $\delta^+(\Xi)$ is a bounded operator on $\Gamma(H)$.*

PROOF. The proof is immediate from Theorem 5.4. $\square$

Remark 5.6 The *classical Hitsuda–Skorohod integral* δ is defined as the adjoint map of the classical stochastic gradient D (see [8, 18, 22, 29]), i.e., for given $\Psi \in L^2(\mathbb{R}_+, \mathcal{G}^*)$, the classical Hitsuda–Skorohod integral $\delta(\Psi) \in \mathcal{G}^*$ of Ψ is defined by

$$\langle\langle \delta(\Psi),\, \phi \rangle\rangle = \int_{\mathbb{R}_+} \langle\langle \Psi(t),\, D_t\phi \rangle\rangle \, dt, \qquad \phi \in \mathcal{G}. \tag{5.3}$$

Therefore, by denoting $(\Xi\phi)(t) = \Xi(t)\phi$, from (5.2) we have

$$\delta^+(\Xi)\phi = \delta\left(\Xi\phi\right), \qquad \phi \in \mathcal{G}, \tag{5.4}$$

see [2, 24, 18].

The creation and annihilation integrals are related directly. The following corollary gives a relation between creation and annihilation integrals.

Corollary 5.7 ([18]) *Let $p, q \in \mathbb{R}$ be given and $\Xi \in L^2(\mathbb{R}_+, \mathcal{L}(\mathcal{G}_p, \mathcal{G}_q))$ be a quantum stochastic process. Then it holds that*

$$(\delta^-(\Xi))^* = \delta^+(\Xi^*). \tag{5.5}$$

PROOF. For any $\phi, \psi \in \mathcal{G}$, we obtain that

$$\begin{aligned}\left\langle\!\left\langle \delta^-(\Xi)\phi,\, \psi \right\rangle\!\right\rangle &= \int_{\mathbb{R}_+} \langle\langle \Xi(t)(D_t\phi),\, \psi \rangle\rangle \, dt = \int_{\mathbb{R}_+} \langle\langle \Xi^*(t)\psi,\, (D_t\phi) \rangle\rangle \, dt \\ &= \left\langle\!\left\langle \delta^+(\Xi^*)\psi,\, \phi \right\rangle\!\right\rangle,\end{aligned}$$

which proves (5.5). $\square$

Theorem 5.8 *Let $p, q \in \mathbb{R}$ be given and $\Xi \in L^\infty(\mathbb{R}_+, \mathcal{L}(\mathcal{G}_p, \mathcal{G}_q))$ be a quantum stochastic process. Then there exists an admissible operator, denoted by $\delta^0(\Xi)$, in $\mathcal{L}(\mathcal{G}_{p+r}, \mathcal{G}_{q-r})$ for any $r > 0$ such that*

$$\left\langle\!\left\langle \delta^0(\Xi)\phi,\, \psi \right\rangle\!\right\rangle = \int_{\mathbb{R}_+} \langle\langle \Xi(t)D_t\phi,\, D_t\psi \rangle\rangle \, dt \tag{5.6}$$

for $\phi, \psi \in \mathcal{G}$.

PROOF. The proof is a simple modification of the proofs of Theorems 5.1 and 5.4. $\square$

For given $\Xi \in L^2(\mathbb{R}_+, \mathcal{L}(\mathcal{G}_p, \mathcal{G}_q))$, the admissible operator $\delta^0(\Xi)$ satisfying (5.6) is called the *conservation integral* of Ξ, see [3, 24, 16, 18].

As for a criterion for $\delta^0(\Xi)$ being a bounded operator on $\Gamma(H)$, we have the following corollary. A similar result can be found in [18].

Corollary 5.9 *For any $r > 0$ and $\Xi \in L^2(\mathbb{R}, \mathcal{L}(\mathcal{G}_{-r}, \mathcal{G}_r))$, the conservation integral $\delta^0(\Xi)$ is a bounded operator on $\Gamma(H)$.*

PROOF. The proof is immediate from Theorem 5.8. $\square$

5.2 Extensions of Anticipating Quantum Stochastic Integrals

In this section, motivated by the results in [23], we discuss extensions of the quantum Hitsuda-Skorohod integrals studied in Section 5.1. Based on the quantum white noise calculus [12], we have the following integral representations:

$$A_t = \int_0^t a_s ds, \quad A_t^* = \int_0^t a_s^* ds, \quad \Lambda_t = \int_0^t a_s^* a_s ds,$$

where a_t and a_t^* are the pointwisely defined annihilation and creation operators. On the other hand, the pointwisely defined annihilation operator a_t and the stochastic gradient D_t coincide on a certain domain. Hence, the following informal computations gives motivations for extensions of the quantum Hitsuda-Skorohod integrals: for a given quantum stochastic process $\{\Xi_t\}_{t\geq 0} \subset \mathcal{L}(\mathcal{G}, \mathcal{G}^*)$ of enough regular operators Ξ_t, we may write as

$$\begin{aligned}
\int_0^t \Xi_s dA_s &= \int_0^t \Xi_s D_s ds = \delta^-(\mathbf{1}_{[0,t]}\Xi), \\
\int_0^t (dA_s)\,\Xi_s &= \int_0^t D_s \Xi_s ds = \int_0^t \Xi_s D_s ds + \int_0^t D_s^+ \Xi_s ds = \delta^-(\mathbf{1}_{[0,t]}\Xi) + \int_0^t D_s^+ \Xi_s ds, \\
\int_0^t (dA_s^*)\,\Xi_s &= \int_0^t D_s^* \Xi_s ds = \delta^+(\mathbf{1}_{[0,t]}\Xi), \\
\int_0^t \Xi_s dA_s^* &= \int_0^t \Xi_s D_s^* ds = \delta^+(\mathbf{1}_{[0,t]}\Xi) + \int_0^t D_s^- \Xi_s ds, \\
\int_0^t \Xi_s d\Lambda_s &= \int_0^t \Xi_s D_s^* D_s ds = \delta^0(\mathbf{1}_{[0,t]}\Xi) + \delta^-(\mathbf{1}_{[0,t]} D_\cdot^- \Xi), \\
\int_0^t (d\Lambda_s)\,\Xi_s &= \int_0^t D_s^* D_s \Xi_s ds = \delta^0(\mathbf{1}_{[0,t]}\Xi) + \delta^+(\mathbf{1}_{[0,t]} D_\cdot^+ \Xi).
\end{aligned} \tag{5.7}$$

However, $D_t^{\pm}\Xi_t$ has no meaning directly. For example, we consider the annihilation process $A_t = a(\mathbf{1}_{[0,t]})$ and then

$$D_t^- A_s = \mathbf{1}_{[0,s]}(t).$$

But the annihilation process A_t can be defined as $a(\mathbf{1}_{[0,t)})$ and then we would have $D_t^- A_s = \mathbf{1}_{[0,s)}(t)$. Therefore, $D_t^- A_t$ cannot be defined in a unique way [23]. From the above example, if we deal with quantum stochastic processes, then it is natural to consider two kinds of pointwisely defined annihilation derivative, $D_{t+}^{\pm}$ and $D_{t-}^{\pm}$. Let $\{\Xi_t\}_{s\geq 0} \subset \mathcal{L}(\mathcal{G}, \mathcal{G}^*)$ be a quantum stochastic process. We define

$$D_{t+}^{\pm}\Xi_t = \lim_{s\downarrow t} D_s^{\pm}\Xi_t, \qquad D_{t-}^{\pm}\Xi_t = \lim_{s\uparrow t} D_s^{\pm}\Xi_t,$$

if the limits exist.

Definition 5.10 Let $\{\Xi_t\}_{t\geq 0} \subset \mathcal{L}(\mathcal{G}, \mathcal{G}^*)$ be a quantum stochastic process.

(1) Suppose that $\delta^+(\Xi)$ exists, and $D_{t+}^- \Xi_t$ exists and it is integrable on $\mathbb{R}_+$. Then we define

$$\int_{\mathbb{R}_+} \Xi_t dA_{t+}^* = \delta^+(\Xi) + \int_{\mathbb{R}_+} D_{t+}^- \Xi_t dt. \tag{5.8}$$

(2) Suppose that $\delta^+(\Xi)$ exists, and $D^-_{t-}\Xi_t$ exists and it is integrable on $\mathbb{R}_+$. Then we define

$$\int_{\mathbb{R}_+} \Xi_t dA^*_{t-} = \delta^+(\Xi) + \int_{\mathbb{R}_+} D^-_{t-}\Xi_t dt. \tag{5.9}$$

(3) Suppose that $\delta^+(\Xi)$ exists, and $D^-_{t+}\Xi_t, D^-_{t-}\Xi_t$ exist as integrable functions on $\mathbb{R}_+$. Then we define

$$<\boldsymbol{\alpha}> \int_{\mathbb{R}_+} \Xi_t \circ dA^*_t = \delta^+(\Xi) + \alpha_1 \int_{\mathbb{R}_+} D^-_{t+}\Xi_t dt + \alpha_2 \int_{\mathbb{R}_+} D^-_{t-}\Xi_t dt \tag{5.10}$$

for $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{R}^2$, which is called the $<\boldsymbol{\alpha}>$*-creation integral*.

Theorem 5.11 *Let $p, q \in \mathbb{R}$ be given and let $\Xi \in L^2(\mathbb{R}_+, \mathcal{L}(\mathcal{G}_p, \mathcal{G}_q))$ be a quantum stochastic process.*

(1) *Suppose that $D^-_{t+}\Xi_t$ exists and $D^-_{\cdot+}\Xi_\cdot \in L^1(\mathbb{R}_+, \mathcal{L}(\mathcal{G}_p, \mathcal{G}_{q-r}))$ for some $r > 0$. Then the integral $\int_{\mathbb{R}_+} \Xi_t dA^*_{t+}$ exists as an operator in $\mathcal{L}(\mathcal{G}_p, \mathcal{G}_{q-r})$.*

(2) *Suppose that $D^-_{t-}\Xi_t$ exists and $D^-_{\cdot-}\Xi_\cdot \in L^2(\mathbb{R}_+, \mathcal{L}(\mathcal{G}_p, \mathcal{G}_{q-r}))$ for some $r > 0$. Then the integral $\int_{\mathbb{R}_+} \Xi_t dA^*_{t-}$ exists as an operator in $\mathcal{L}(\mathcal{G}_p, \mathcal{G}_{q-r})$.*

Proof. (1) Since $\Xi \in L^2(\mathbb{R}_+, \mathcal{L}(\mathcal{G}_p, \mathcal{G}_q))$, by Theorem 5.4, the quantum Hitsuda-Skorohod creation integral $\delta^+(\Xi)$ of Ξ exists as an admissible operator in $\mathcal{L}(\mathcal{G}_p, \mathcal{G}_{q-s})$ for any $s > 0$. Also, since, by assumption, $D^-_{t+}\Xi_t$ exists and $D^-_{\cdot+}\Xi_\cdot \in L^1(\mathbb{R}_+, \mathcal{L}(\mathcal{G}_p, \mathcal{G}_{q-r}))$ for some $r > 0$, for any $\phi \in \mathcal{G}$ we obtain that

$$\left\| \int_{\mathbb{R}_+} D^-_{t+}\Xi_t \phi \, dt \right\|_{q-r} \leq \left(\int_{\mathbb{R}_+} \| D^-_{t+}\Xi_t \|_{p;q-r} \, dt \right) \| \phi \|_p ,$$

which implies that the integral $\int_{\mathbb{R}_+} D^-_{t+}\Xi_t \, dt$ exists as an admissible operator in $\mathcal{L}(\mathcal{G}_p, \mathcal{G}_{q-r})$. Finally, the integral $\int_{\mathbb{R}_+} \Xi_t dA^*_{t+}$ exists as an operator in $\mathcal{L}(\mathcal{G}_p, \mathcal{G}_{q-r})$.

(2) The proof is similar to the proof of (1). □

By similar arguments used in Definition 5.10, we can define the quantum stochastic integrals of types given as in (5.7) of which the study will be appear in some other papers.

Acknowledgements This work was supported by Basic Science Research Program through the NRF funded by the MEST (NRF-2016R1D1A1B01008782).

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